

Order \hbar Corrections to the Classical Dynamics of a Particle with Intrinsic Spin Moving in a Constant Magnetic Field

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Abstract

$O(\hbar)$ effects that modify the classical orbit of a charged particle are described for the case of a classical spin- $\frac{1}{2}$ particle moving in a constant magnetic field, using a manifestly covariant formalism reported previously. It is found that the coupling between the momentum and spin gives rise to a shift in the cyclotron frequency, which is explicitly calculated. In addition the orbit is found to exhibit $O(\hbar)$ oscillations along the axis of the uniform static magnetic field whenever the gyromagnetic ratio g of the particle is not 2. This oscillation is found to occur at a frequency $\propto \frac{g}{2} - 1$, and is an observable source of electric dipole radiation.

I. INTRODUCTION

Closed orbit correction is a critical component of background reduction in high energy electron and proton accelerator experimental detectors. In a collider ring, tight quality control of the closed orbit is always essential to the efficient operation of the detectors.

Higher-order non-linear relativistic processes may measurably effect the dynamical trajectory of a classical spin- $\frac{1}{2}$ electron in an electron accelerator or a proton in a particle beam. Such effects can in principle be seen after many orbits in a long-lived charged particle beam in a storage ring when conditions for resonance are met. For example, a key issue in collider physics is to what extent the interaction of non-vanishing magnetic field gradients along the particle's trajectory with the intrinsic magnetic dipole moment of the particle drives the particle from the ideal design orbit (the relativistic Stern-Gerlach effect). This may alter the depolarizing resonance strengths and widths, and may induce new resonant modes that effect the spin. Hence a quantitative description of the contribution of non-linear relativistic effects to closed orbit control is desirable. After over fifty years of study, this subject continues to be an active research topic [4] - [6]. Failure to study these processes could lead to a serious deficiency in our knowledge of the details of high-energy classical dynamics. For example, small innocuous non-linear terms can sometimes lead to spectacular observable experimental consequences (recall the hysteresis in cyclotron resonance based on a weak non-linear relativistic mass effect).

Moreover the recent questions concerning the origin of proton spin (Ashman, *et al* [7], Close and Roberts [8], Ellis and Karliner [9], Meng Ta-chung, *et al* [10]) and the observation of a strong spin-dependence of high-energy proton-proton interactions underline the need for a complete understanding of classical spin.

In this paper we consider the classical motion of a charged particle with intrinsic spin orbiting in a circle about a constant magnetic field normal (in the *zeroth* approximation) to the plane of the orbit. Such fields occur in an ideal accelerator or are approximated by the stellar magnetic field near a pulsar. We study the dynamical problem classically to $O(\hbar)$

and derive several new experimentally testable predictions of classical dynamics. We also clarify a classic result, which states that the ratio of the spin precession angular frequency to the cyclotron frequency is given by $1 + \gamma(\frac{g}{2} - 1)$ [1], [2], [3], where g is the gyromagnetic ratio of the particle and $\gamma^{-2} = 1 - \frac{v^2}{c^2}$. It is shown that this is a time-averaged result, and derive the time-dependent expression in Equation[3.17]. We shall adopt a classical description throughout this paper so that we may describe the electron with the same formalism as the proton. A quantum mechanical self-consistent description of an electron moving in the prescribed fields would require including radiative corrections to the electron-photon vertex in order to account for the fact that $g \neq 2$ for the electron. (Not so for a quantum mechanical description of protons, where one does not hesitate to introduce a phenomenological gyromagnetic ratio.) In the classical description, g is a parameter that one sets equal to an experimentally determined value. Although a classical description is many times inadequate, in this case we shall see that the classical description predicts and describes some effects that are difficult to calculate using relativistic quantum mechanics.

Although not treated in this paper, the method employed to formulate this dynamical problem can be extended to handle the case of charged-particle beams. This is important since the actual synchrotron beam is comprised of multiple bunches of charged particles, with up to 10^{11} particles per beam bunch. The dynamics of the beam is determined by the applied external fields plus the electromagnetic fields generated by the particles in the interacting bunches. The normalized precessional frequency is shifted by the beam-beam interaction and these shifts can also be approximated.

II. EQUATIONS OF MOTION

The Lorentz force equations for the four-velocity $\frac{dx^\alpha}{d\tau} \stackrel{\text{def}}{=} \dot{x}^\alpha$ coupled with the Thomas-Frenkel-Bargmann-Michael-Telegdi (BMT) equations for the dynamical evolution of the Pauli-Lubanski spin vector govern the motion of a polarized particle in the limit $\hbar \rightarrow 0$. Here τ is the proper time. These equations have been extended to higher order in \hbar . Al-

though there are many specific models ^{[11]–[18]}, we shall employ the dynamical equations derived by Nash ^[19]. This coupled set of equations reduces *exactly* to the standard BMT and Lorentz force equations in the limit $\hbar \rightarrow 0$. Moreover these equations are derived from a Lagrangian using a variational principle, which ensures that the dynamical equations do not conflict with conservation laws. We turn briefly to a summary of the approach of Ref [19].

Units are used in which the speed of light is one. The Minkowski spacetime metric is $\eta_{\alpha\beta} = \text{diag}(1, 1, 1, -1)$. The dynamical variables are the particle's position x^α , $\alpha = 1, \dots, 4$ in spacetime and ψ , which is a real eight-component spinor. ψ may be regarded as a column vector comprised of the direct sum of a real four-component Dirac spinor λ [transforming under the *real* irreducible representation of $\overline{SO(3,3)}$ defined by Dirac (see Ref [20])] and the transpose of another real four-component Dirac spinor ξ that transforms under the inverse irreducible representation

$$\psi = \begin{pmatrix} \lambda \\ \tilde{\xi} \end{pmatrix}, \quad (2.1)$$

where the tilde denotes the transpose. ψ defines an orthonormal tetrad in Minkowski spacetime ^[21]. The members of the tetrad are constructed from a sum of products of the components of ψ . It is known ^[22] that ψ may be regarded as a (split) octonian.

Explicitly, the timelike member of the tetrad is given by

$$E_{(4)}^\alpha(\tau) = -\frac{1}{2}\tilde{\psi}\Gamma^4\Gamma^\alpha\psi, \quad (2.2)$$

where the Γ_α matrices are real 8×8 analogs of Dirac's gamma matrices, and is parallel to the four-velocity \dot{x}^α when Equation [2.4] (below) is satisfied. In passing, we record for later use the definition of the third member of the tetrad

$$E_{(3)}^\alpha(\tau) = \frac{1}{2}\tilde{\psi}\Gamma^4\Gamma^7\Gamma^\alpha\psi. \quad (2.3)$$

For simplicity of presentation it is assumed that ψ has been normalized to $\tilde{\psi}\psi = \dot{x}^4$. The general case is obtained by dividing by a normalizing factor proportional to $\xi\gamma^5\lambda$.

The classical set of dynamical variables $\{x^\alpha, \psi\}$ are not all independent. This is because the timelike $E_{(4)}^\alpha$ constructed from an arbitrary ψ is in general not parallel to \dot{x}^α . But in the free-field case one knows that only one timelike vector is required for the description of particle ‘dynamics.’ In order for $\{x^\alpha, \psi\}$ to describe a massive particle with spin, ψ must be constrained so that $E_{(4)}^\alpha$ and \dot{x}^α are parallel. The crucial result we shall use is that the timelike member of the tetrad can be permanently aligned with the four-velocity \dot{x}^α of the particle by imposing the constraint ^[19]

$$\left(\Gamma_\alpha \dot{x}^\alpha + \sqrt{-\dot{x}_\alpha \dot{x}^\alpha} \Gamma^7\right) \psi = 0. \quad (2.4)$$

This *classical* constraint bears a striking resemblance to the quantum mechanical Dirac equation. This suggests that one may quantize this theory by “quantizing the constraint” and postulating minimal coupling. However we do not treat the problem considered in Section III quantum mechanically. To do so one must consistently take into account higher order corrections to the magnetic moment of the particle that arise from radiative corrections to the lepton-photon vertex operator. In the classical approach one may use the experimentally measured gyromagnetic ratio in the classical equations of motion. One obtains an approximation for the particle’s orbit.

The constraint Equation [2.4] is incorporated into the theory using a Lagrange multiplier. If this constraint is satisfied then the third member of the tetrad may be identified with the Pauli-Lubanski spin vector, and moreover, the intrinsic electric dipole moment of the particle vanishes in a rest frame $\Sigma_{\alpha\beta} \dot{x}^\beta = 0$, where $\Sigma^{\alpha\beta}$ is the spin tensor of the particle defined by

$$\Sigma^{\alpha\beta} = -\frac{1}{2} \tilde{\psi} \Omega \left(M^{\alpha\beta}\right) \psi. \quad (2.5)$$

Here Ω is the symplectic form on the spinor manifold and

$$M^{\alpha\beta} = -\frac{1}{4} [\Gamma^\alpha, \Gamma^\beta]. \quad (2.6)$$

The Lagrangian for the theory is

$$\mathcal{L} = -M \sqrt{-\dot{x}_\alpha \dot{x}^\alpha} - \frac{\hbar}{2} \tilde{\psi} \Omega \dot{\psi} + e A_\alpha \dot{x}^\alpha + \text{Lagrange multiplier term} \quad (2.7)$$

where e is the charge of the particle, A_α is the electromagnetic vector potential, and M denotes the effective mass of the particle, which takes into account the spin-field interaction energy contribution to the particle's mass. It is given by

$$M = m \sqrt{1 - \frac{ge\hbar}{2m^2} \Sigma^{\alpha\beta} F_{\alpha\beta}} \quad (2.8)$$

where m is the rest mass of the particle (or simply a parameter with dimension of mass, which is to be renormalized to the observed rest mass) and $F_{\alpha\beta}$ is the electromagnetic field tensor. $F_{\alpha\beta}$ contains a contribution from the radiation reaction of the particle. If this contribution to $F_{\alpha\beta}$ is completely neglected then the equations of motion admit unphysical solutions in the free-field limit.

The canonical four-momentum p_α is given by

$$p_\alpha = Mx_\alpha + eA_\alpha + \hbar \Sigma_{\alpha\beta} b^\beta \quad (2.9)$$

. the classical version The equations of motion derived from this Lagrangian are

$$\hbar \dot{\psi} = -\hbar M^{\alpha\beta} \psi \left(\frac{ge}{4M} F_{\alpha\beta} + \dot{x}_\alpha b_\beta \right) \quad (2.10)$$

where

$$b^\alpha = \ddot{x}^\alpha - \frac{ge}{2M} F^\alpha_\beta \dot{x}^\beta. \quad (2.11)$$

Also

$$M\ddot{x}^\alpha = eF^\alpha_\beta \dot{x}^\beta + P^{\alpha\mu} \Upsilon_\mu, \quad (2.12)$$

where $P^{\alpha\mu}$ is a projection operator $P^{\alpha\mu} = \eta^{\alpha\mu} + \dot{x}^\alpha \dot{x}^\mu$ and $\Upsilon_\mu = \frac{ge\hbar}{4M} \Sigma^{\beta\lambda} \frac{\partial F_{\beta\lambda}}{\partial x^\mu} - \frac{d}{d\tau} (\hbar \Sigma_{\mu\beta} b^\beta)$ contains the contribution to the force due to the Stern-Gerlach effect.

For completeness we mention that upon substituting the result Equation [2.10] into the definition of the tetrad $E_{(j)}^\alpha$ yields

$$\dot{E}_{(j)}^\alpha = \frac{ge}{2M} F^\alpha_\beta E_{(j)}^\beta + \dot{x}^\alpha b_\beta E_{(j)}^\beta. \quad (2.13)$$

where $j = 1, 2, 3$ labels the spacelike members of the tetrad. It is worth noting that in the limit $\hbar \rightarrow 0$ Equation [2.12] is the Lorentz force equation and Equation [2.13], with $j = 3$, is exactly the BMT equation.

III. UNIFORM STATIC VERTICAL MAGNETIC FIELD

In order to simplify the notation we define two sets of three 8×8 matrices

$$\sigma = (M^{23}, M^{31}, M^{12}) \quad (3.1)$$

and

$$\mathbf{m} = (M^{14}, M^{24}, M^{34}). \quad (3.2)$$

We employ the representation of the Γ matrices defined in References [22] and [23].

The electromagnetic field tensor $F_{\alpha\beta}$ is represented in terms of the electric field \mathbf{E} and magnetic field \mathbf{B} as

$$F_{\alpha\beta} = \begin{pmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix} \quad (3.3)$$

We assume that the $\hbar \rightarrow 0$ limit of the Lorentz force equations have been solved so that $x^\alpha = x^\alpha(\tau)$ is known. Substituting this result into the $\frac{d}{d\hbar}\big|_{\hbar \rightarrow 0}$ limit of Equation [2.10] yields

$$\dot{\psi} = V(\tau)\psi, \quad (3.4)$$

where

$$V = \frac{e}{2m} \left[-\frac{g}{2} F_{\alpha\beta} + \left(\frac{g}{2} - 1 \right) \dot{x}^\mu (\dot{x}_\alpha F_{\beta\mu} - \dot{x}_\beta F_{\alpha\mu}) \right] M^{\alpha\beta} \quad (3.5)$$

is real 8×8 matrix that is a known function of tau.

In the notation introduced above Equation [3.5] reduces to

$$\begin{aligned} \frac{m}{e} V = & -\left\{ 1 + \gamma^2 \left(\frac{g}{2} - 1 \right) \right\} \sigma \cdot \mathbf{B} + \left(\frac{g}{2} - 1 \right) \{ \dot{\mathbf{x}} \cdot \mathbf{B} \} \sigma \cdot \dot{\mathbf{x}} \\ & - \gamma \left(\frac{g}{2} - 1 \right) \mathbf{m} \cdot \mathbf{B} \times \dot{\mathbf{x}} - \left(\frac{g}{2} - 1 \right) \{ \dot{\mathbf{x}} \cdot \mathbf{E} \} \mathbf{m} \cdot \dot{\mathbf{x}} \\ & - \gamma^2 \left(1 - \beta^2 \frac{g}{2} \right) \mathbf{m} \cdot \mathbf{E} + \gamma \left(\frac{g}{2} - 1 \right) \sigma \cdot \dot{\mathbf{x}} \times \mathbf{E}. \end{aligned} \quad (3.6)$$

For the limiting case $\hbar \rightarrow 0$, $\mathbf{B} = \hat{\mathbf{z}}B_3$, $B_3 = \text{constant}$, the electromagnetic field has only one non-vanishing independent component in the lab frame. This is $F_{12} = B_3$. The Lorentz force equations [2.12] are independent of ψ to $zero^{th}$ order in \hbar . The $zero^{th}$ order solution $\hbar \rightarrow 0$ is of course well known. For a circular orbit in the $\mathbf{x} \wedge \mathbf{y}$ plane a solution to the Lorentz force equation is $\dot{x}^1 = \beta\gamma \cos(\omega_0\tau)$, $\dot{x}^2 = -\beta\gamma \sin(\omega_0\tau)$, $\dot{x}^3 = 0$ and $\dot{x}^4 = \gamma = \text{constant}$, where $\omega_0 = \frac{eB_3}{m}$, $\beta = \frac{v}{c} = v = \text{constant}$ and $\gamma^2 = \frac{1}{1-\beta^2}$. The cyclotron frequency is $\omega_{cyclotron} = |\omega_0| / \gamma$.

If we put $T = \omega_0\tau$ and $\nu = \gamma(\frac{g}{2} - 1)$, and use Equation [3.6], then Equation [3.4] yields

$$\frac{d\psi}{dT} = -\{(1 + \gamma\nu)M^{12} + \beta\gamma\nu[\sin(T)M^{14} + \cos(T)M^{24}]\}\psi. \quad (3.7)$$

Upon making the substitution

$$\phi = \exp[(1 + \gamma\nu)TM^{12}]\psi, \quad (3.8)$$

and simplifying, one finds that ϕ satisfies

$$\begin{aligned} \frac{d\phi}{dT} &= -\beta\gamma\nu[-\sin(\gamma\nu T)M^{14} + \cos(\gamma\nu T)M^{24}]\phi \\ &= -\beta\gamma\nu e^{\gamma\nu TM^{12}} M^{24} e^{-\gamma\nu TM^{12}} \phi, \end{aligned} \quad (3.9)$$

where we have used

$$\begin{aligned} e^{(1+\gamma\nu)TM^{12}} [\sin(T)M^{14} + \cos(T)M^{24}] e^{-(1+\gamma\nu)TM^{12}} \\ = -\sin(\gamma\nu T)M^{14} + \cos(\gamma\nu T)M^{24}. \end{aligned} \quad (3.10)$$

The general solution to Equation [3.9] is now easily found. One finds that $\psi = \Lambda_{\mathbf{B}_3, g, \beta, \tau} \psi(0)$, where the constants of integration are in $\psi(0)$ and

$$\begin{aligned} \Lambda_{\mathbf{B}_3, g, \beta, \tau} &= e^{-TM^{12}} e^{-\gamma\nu T(M^{12} + \beta M^{24})} \\ &= e^{-\kappa[\cos(T)M^{14} - \sin(T)M^{24}]} e^{-(1+\nu)TM^{12}} e^{\kappa M^{14}}, \end{aligned} \quad (3.11)$$

and $\kappa = \frac{1}{2} \ln\{\frac{1+\beta}{1-\beta}\}$. e^κ is Bondi's 'k' factor [24]. $\Lambda_{\mathbf{B}_3, g, \beta, \tau}$ is decomposed into the product of a boost in the direction of $-\dot{\mathbf{x}}(0)$, times a τ -dependent rotation about the guiding magnetic

field, times another boost in the $\hat{\mathbf{x}}\cos(T) - \hat{\mathbf{y}}\sin(T) \propto \dot{\mathbf{x}}(T)$ direction. $\Lambda_{\mathbf{B}_3, g, \beta, \tau}$ commutes with Γ^3 and hence does not affect the polarization Σ_3 .

In order to calculate the Pauli-Lubanski spin vector from $\psi(\tau) = S_0\psi(0)$, with $S_0 = \Lambda_{\mathbf{B}_3, g, \beta, \tau}$ we first solve the eigenvalue problem Equation [2.4] for $\psi(0)$, choosing initial phases that simplify our work:

$$\left(\Gamma_\alpha \dot{x}^\alpha(0) + \sqrt{-\dot{x}_\alpha(0)\dot{x}^\alpha(0)} \Gamma^7 \right) \psi(0) = 0. \quad (3.12)$$

We find that there are four linearly independent solutions to this eigenvalue problem for $\dot{x}^4 > 0$, two solutions for spin up, and two solutions with spin down. As a check on our work we substitute $\psi(0)$ and S_0 into the definition $E_{(4)}^\alpha(\tau) = -\frac{1}{2}\tilde{\psi}(0)\tilde{S}_0\Gamma^4\Gamma^\alpha S_0\psi(0)$. Each distinct spinor solution, when substituted into this equation yields the four-velocity $E_{(4)}^\alpha = (\beta\gamma\cos(T), -\beta\gamma\sin(T), 0, \gamma) = \dot{x}^\alpha$, as required. Next, substitution of the four spinor eigenvectors in turn into the definition Equation [2.3] yields two distinct Pauli-Lubanski four-vectors, which differ only in the sign of the third component. For example, we find that a “spin up” spinor eigenvector is

$$\tilde{\psi}(0)_\uparrow = (0, 0, 1, 0, 0, -\beta\gamma, 0, \gamma)/\sqrt{2\gamma}, \quad (3.13)$$

and $\psi = \Lambda_{\mathbf{B}_3, g, \beta, \tau} \psi(0)_\uparrow$ is given by

$$\psi = \frac{1}{\sqrt{2\gamma}} \begin{pmatrix} -\beta\gamma \sin(T/2) \sin(\nu T/2) \\ -\beta\gamma \cos(T/2) \sin(\nu T/2) \\ \cos(T/2) \cos(\nu T/2) - \gamma \sin(T/2) \sin(\nu T/2) \\ -\sin(T/2) \cos(\nu T/2) - \gamma \cos(T/2) \sin(\nu T/2) \\ -\beta\gamma \sin(T/2) \cos(\nu T/2) \\ -\beta\gamma \cos(T/2) \cos(\nu T/2) \\ \cos(T/2) \sin(\nu T/2) + \gamma \sin(T/2) \cos(\nu T/2) \\ -\sin(T/2) \sin(\nu T/2) + \gamma \cos(T/2) \cos(\nu T/2) \end{pmatrix} \quad (3.14)$$

The Pauli-Lubanski spin four-vector in this case is

$$\begin{aligned}
E_{(3)}^\alpha &= \left(-\frac{1}{2}\beta[(\gamma+1)\cos(\nu+1)T + (\gamma-1)\cos(\nu-1)T], \right. \\
&\quad \left. \frac{1}{2}\beta[(\gamma+1)\sin(\nu+1)T - (\gamma-1)\sin(\nu-1)T], \right. \\
&\quad \left. \gamma^{-1}, -\beta^2\gamma\cos\nu T\right). \tag{3.15}
\end{aligned}$$

$E_{(3)}^\alpha$ is just the Lorentz transform to the lab frame of the spin 3-vector $\mathbf{s} = (-\beta\cos[(\nu+1)T], \beta\sin[(\nu+1)T], \frac{1}{\gamma})$, which may be verified by transforming $E_{(3)}^\alpha$ to an instantaneous rest frame with the Lorentz boost L_{boost} defined as usual by $x'^i = x'^j(\delta_j^i - n^i n_j + n^i n_j) = (\delta_j^i - n^i n_j)x^j + n^i\gamma(\hat{\mathbf{n}} \cdot \vec{x} - \beta ct) \equiv L_{boost}^i{}_\alpha x^\alpha$ and $ct' = \gamma(ct - \beta\hat{\mathbf{n}} \cdot \vec{x}) \equiv L_{boost}^4{}_\alpha x^\alpha$, where n^i is a unit vector parallel to the 3-velocity of the particle. Applying this boost to $E_{(3)}^\alpha$ yields

$$E_{(3)}^{rest\ frame\ \alpha} = (-\beta\cos[(\nu+1)T], \beta\sin[(\nu+1)T], \frac{1}{\gamma}, 0). \tag{3.16}$$

Since $\dot{x}_\alpha E_{(3)}^\alpha = 0$, the angle θ_R between the three-velocity \dot{x}^j and the three-spin $E_{(3)}^j$ is νT . One sees that $\theta_R \stackrel{def}{=} \nu T = \nu\omega_0\tau = \gamma(\frac{g}{2} - 1)\frac{eB_3}{m}\frac{t}{\gamma}$, which is the well known result for the precession of the longitudinal polarization [26].

The angular velocity of precession $\omega_{precess}$ that is measured in the lab frame is given by

$$\begin{aligned}
\omega_{precess}/\omega_{cyclotron} &= \left| (E_{(3)}^1 \frac{dE_{(3)}^2}{dT} - E_{(3)}^2 \frac{dE_{(3)}^1}{dT}) / (E_{(3)}^1)^2 + (E_{(3)}^2)^2 \right| \\
&= 1 + \frac{g/2 - 1}{1 - \beta^2 \sin^2(\nu T)}. \tag{3.17}
\end{aligned}$$

The average over a time $\nu T = 2\pi$ is

$$\langle \omega_{precess}/\omega_{cyclotron} \rangle = 1 + \gamma(\frac{g}{2} - 1) = 1 + \nu. \tag{3.18}$$

which is the well-known expression often identified as the normalized spin precessional frequency (see, for example, References [1], [2] and [3]).

In order to write out and solve the momentum equations [2.12] to $O(\hbar)$ we must first evaluate the components of the spin tensor. Substituting ψ from Equation [3.14] into Equation [2.5] yields

$$\begin{pmatrix} \Sigma^{23} \\ \Sigma^{31} \\ \Sigma^{12} \\ \Sigma^{14} \\ \Sigma^{24} \\ \Sigma^{34} \end{pmatrix} = \begin{pmatrix} \beta[-\cos(T)\cos(\nu T) + \gamma\sin(T)\sin(\nu T)] \\ \beta[\sin(T)\cos(\nu T) + \gamma\cos(T)\sin(\nu T)] \\ 1 \\ -\beta\sin(T) \\ -\beta\cos(T) \\ -\beta^2\gamma\sin(\nu T) \end{pmatrix}. \quad (3.19)$$

In terms of their Fourier decompositions, $\Sigma^{23} = \frac{\beta}{2}[-(\gamma+1)\cos((\nu+1)T) - (\gamma-1)\cos((\nu-1)T)]$ and $\Sigma^{31} = \frac{\beta}{2}[(\gamma+1)\sin((\nu+1)T) + (\gamma-1)\sin((\nu-1)T)]$.

A word about normalization. Throughout this paper the normalization of ψ is determined by the arbitrary requirement that the $E_{(\beta)}^\alpha$ comprise a *normalized* set of four mutually orthogonal vectors, which is reflected in the result that $\Sigma^{12} = 1$. This is not a statement about the magnitude of the classical spin. The magnitude of the classical spin is not predicted by this theory. Instead it must be imposed as an initial condition; in virtue of the equations of motion this magnitude is a constant of the motion. Given this magnitude and g we may employ this formalism to compute the trajectory. Conversely, one can impose the observed spin magnitude as an initial condition and then use the predictions of this theory and experimental measurements to determine g . One sees that to correctly apply this dynamical formalism to a spin- s particle one must arrange that $\Sigma^{12} = s$. This is easy to accomplish by imposing this as an initial condition, and is manifested in the simple replacement $\psi(0) \mapsto \sqrt{s}\psi(0)$, which we shall henceforth employ.

We turn now to the $O(\hbar)$ solution of the extended Lorentz equations for the momentum. The effective mass $M = m\sqrt{1 - \frac{ge\hbar}{2m^2}\Sigma^{\alpha\beta}F_{\alpha\beta}} \approx m - s\frac{g}{2}\hbar\omega_0$ to $O(\hbar)$. Also to this order, $\hbar b^\alpha = -G\hbar\omega_0 F_{\beta\ 0}\dot{x}^\beta/B_3$, where $G = \frac{g}{2} - 1$ and ${}_0\dot{x}^\beta$ refers to the $O(1) \stackrel{\text{def}}{=} \hbar \rightarrow 0$ solution to the momentum equations. We find that

$$\hbar \Sigma_{\beta}^{\alpha} b^{\beta} = s\beta\gamma\hbar\omega_0 G \begin{pmatrix} \cos(T) \\ -\sin(T) \\ \beta \cos(\nu T) \\ \beta \end{pmatrix} = s\hbar\omega_0 G \begin{pmatrix} {}_0\dot{x}^1 \\ {}_0\dot{x}^2 \\ \beta^2\gamma \cos(\nu T) \\ \beta^2 {}_0\dot{x}^4 \end{pmatrix}. \quad (3.20)$$

We note that ${}_0\dot{x}^{\alpha}\Sigma_{\alpha\beta}b^{\beta} = 0 = {}_0\dot{x}^{\alpha}\frac{d}{d\tau}(\Sigma_{\alpha\beta}b^{\beta})$.

We write $\dot{x}^{\alpha} = {}_0\dot{x}^{\alpha} + \hbar {}_1\dot{x}^{\alpha}$ and solve $\frac{d}{d\tau}(M\dot{x}^{\alpha}) = eF_{\beta}^{\alpha}\dot{x}^{\beta} - \hbar P^{\alpha\mu}\frac{d}{d\tau}(\Sigma_{\mu\beta}b^{\beta})$. Substituting for $P^{\alpha\mu}$ and rearranging terms yields $\frac{d}{d\tau}(M\dot{x}^{\alpha} + \hbar\Sigma_{\beta}^{\alpha}b^{\beta}) = eF_{\beta}^{\alpha}\dot{x}^{\beta} - \hbar\dot{x}^{\alpha}\dot{x}^{\mu}\frac{d}{d\tau}(\Sigma_{\mu\beta}b^{\beta}) = eF_{\beta}^{\alpha}\dot{x}^{\beta} - \hbar {}_0\dot{x}^{\alpha} {}_0\dot{x}^{\mu}\frac{d}{d\tau}(\Sigma_{\mu\beta}b^{\beta}) = eF_{\beta}^{\alpha}\dot{x}^{\beta}$ since ${}_0\dot{x}^{\alpha}\frac{d}{d\tau}(\Sigma_{\alpha\beta}b^{\beta}) = 0$. For $\alpha = i = 1, 2$ we see that

$$(m - s\hbar\omega_0)\ddot{x}^i = eF_{\beta}^i\dot{x}^{\beta}. \quad (3.21)$$

This is independent of g , but dependent on the magnitude of the spin, and leads to a shift in the cyclotron frequency. This shift is in the other direction for a particle with spin down. For a particle with $s = \frac{1}{2}$ one sees that $\gamma\omega_{cyclotron} = \left|\frac{eB_3}{m - \frac{1}{2}\hbar\omega_0}\right| = \frac{\omega_0}{1 - \frac{1}{2}\frac{\hbar\omega_0}{m}}$. This effect may be of interest for the case of plasma dynamics in critical magnetic fields of the order of $10^{12} - 10^{14} Gauss$, which may exist near pulsars.

Continuing with the analysis we find that $\dot{x}^4 = \gamma = constant$ and

$$\begin{aligned} \dot{x}^3 &= -\beta^2 \frac{\hbar s \omega_0}{m} \nu \cos(\nu T) + constant \\ &= -\frac{d}{d\tau} \left(\beta^2 \frac{\hbar s}{m} \sin(\nu T) \right), \end{aligned} \quad (3.22)$$

where for simplicity we have set the integration constant equal to zero. The charge will radiate in virtue of this oscillation along the axis of the applied magnetic field. The oscillation exists because the four-spin is orthogonal to the four-velocity, and the spin precesses.

IV. CONCLUSION

The theory predicts a contribution to the electric dipole radiation of the particle with frequency $\omega = \omega_0(\frac{g}{2} - 1)$ and power on the order of $\frac{1}{3}ck^4|\vec{p}|^2 = \frac{\omega^4}{6c^3}(\beta^2 e \frac{\hbar}{2mc})^2$ due to the

$O(\hbar)$ oscillations of the orbit along the direction of the applied static magnetic field. (There are of course other $O(\hbar)$ sources of electric dipole radiation that we are not considering here. One such source is proportional to $\hbar\partial_\alpha\frac{\Sigma^{\alpha\beta}}{M}$.) This suggests a new way to measure the gyromagnetic ratio of the electron or proton. One can measure $\frac{g}{2} - 1$ by measuring the frequency of the electric dipole radiation due to the $O(\hbar)$ oscillations of the orbit along the direction of the applied static magnetic field.

Radiation reaction terms have not been included in the analysis. However, radiation damping tends to polarize electrons in high energy accelerators. These effects are observable and must be included in any complete description of relativistic dynamics of light particles. The radiation damping force is usually assumed to arise from the self-field of the accelerated charged particle *via* the Lorentz force. Dirac has shown that the finite (on the world line of the particle) contribution to the self-field is given by $F_{rad}^{\alpha\beta} = \frac{1}{2} (F_{ret}^{\alpha\beta} - F_{adv}^{\alpha\beta}) = \frac{2e}{3} (v^\alpha \ddot{v}^\beta - v^\beta \ddot{v}^\alpha) = \frac{2e}{3} \frac{d}{d\tau} (v^\alpha \dot{v}^\beta - v^\beta \dot{v}^\alpha)$, where $v^\alpha = \dot{x}^\alpha$. Adding this field to the externally applied electromagnetic field $F^{\alpha\beta}$ in Equation [2.12] yields a generalization of the Abraham-Lorentz equation. Such equations are well known to be plagued with “self accelerated” runaway solutions. One simple way around this is to replace \dot{v}^α with $\frac{e}{M} F^\alpha_\beta v^\beta$, since an external $F^{\alpha\beta}$ is ultimately responsible for $\dot{v}^\alpha \neq 0$. This yields a second order ODE for x^α (coupled to the first order ODE for ψ).

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